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ON THE RELATIVISTIC OSCILLATOR

Z. Reut
Chancery House, 319 *City Road*, *London EC*1*V* 1*LJ*, *England*

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The relativistic oscillator is substantially more complex to solve than the classical oscillator because the differential equations of motion are non-linear. The oscillatory motion can be non-linear in classical systems as, for example, in the case of a stretched string with a midpoint mass [1], and it is of interest to estimate the non-linearity present due to the relativistic nature of motion, in particular, in view of the possibility of increasingly more accurate measurements. The relativistic oscillator can be considered as a special case of the relativistic motion of a particle in a central field of force, providing that the force is directly proportional to the distance from the centre of force. The theory of relativistic motion can be represented in classical space and time as in the Newtonian theory by introducing a generalized velocity-dependent force, which contains a constant parameter equal to the speed of light [2, 3]. In the case of an inverse-square field of force, the solution can be found in terms of elementary functions, but that does not apply to the case of a direct-distance field of force. If the initial velocity of the particle is in the direction of the centre of force, the motion is rectilinear and the differential equation of motion is

$$m\ddot{x} + kx(1 - \dot{x}^2/c^2)^{3/2} = 0, \tag{1}$$

where $\dot{x}$ and $\ddot{x}$ are the first and the second order derivatives of x with respect to time, m is the rest mass of the particle, k is the coefficient of restitution of the force, and c is the speed of light in a vacuum. Since the angular momentum with respect to the centre of force is equal to zero, the case of rectilinear motion has to be treated separately, rather than as a particular case of planar motion. Equation (1), which includes the effect of longitudinal mass increase, was adopted by other authors as the differential equation of relativistic one-dimensional oscillator; with a rescaling of the variables it can be written in a form obtained by setting the parameters m, k, and c equal to unity, and a periodic solution can be found by a transformation of the variables and the method of harmonic balance [4]. However, since the exactness of the method of harmonic balance remains in question, it can be advantageous to find an approximate solution, as accurate as possible in terms of elementary functions, by the classical method.

The first integral of motion obtained by the integration of equation (1) that represents the conservation of energy in the form

$$mc^2(1 - \dot{x}^2/c^2)^{-1/2} + \tfrac{1}{2}kx^2 = E, \tag{2}$$

where E is the constant of integration equal to the relativistic total energy of the particle, inclusive of the rest energy, in the oscillator potential. Equation (2) can be solved

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algebraically for the derivative $\dot{x}$ and gives

$$\dot{x} = \pm\, c\{1 - [mc^2/(E - \tfrac{1}{2}kx^2)]^2\}^{1/2}, \tag{3}$$

where the two signs correspond to two possible directions of motion.

The extrema values of the variable x are determined by instants where the derivative $\dot{x}$ vanishes; then equation (3) with the left-hand side set equal to zero can be solved algebraically for the variable x, giving for the distance extremum

$$x_e = \pm\, [2(E - mc^2)/k]^{1/2}, \tag{4}$$

where the two signs correspond to two extrema of the motion for a given relativistic total energy E.

The extrema values of the derivative $\dot{x}$ are given by equation (3) for $x = 0$, that is, they are at the equilibrium position. The limits as the derivative extrema approach $\pm c$ correspond to an infinite relativistic total energy E.

The final equation of motion $x = x(t)$ should follow by an integration of equation (3), but it is not reducible to elementary functions. Upon introducing the variable $y = mc^2/(E - \frac{1}{2}kx^2)$, equation (3) takes the form

$$\dot{y} = \pm\, 2^{1/2}\omega y^{3/2}(ey - 1)^{1/2}(1 - y)^{1/2}(1 + y)^{1/2}, \tag{5}$$

where $\dot{y}$ on the left-hand side is the first-order derivative of y with respect to time, $\omega = (k/m)^{1/2}$, and $e = E/mc^2 > 1$, since the oscillator potential is always positive. The range of the variable is $1/e < y < 1$. In the case of low energy, e is close to unity and the range of the variable is limited to the vicinity of one; in the ultrarelativistic limit, e increases to infinity and the range expands to the interval (0, 1).

Upon introducing the variable $z = (ey - 1)/(e - 1)$, equation (5) becomes

$$\dot{z} = \pm\, 2^{1/2}\omega e^{-3/2}[1 + (e - 1)z]^{3/2}[e + 1 + (e - 1)z]^{1/2}z^{1/2}(1 - z)^{1/2}, \tag{6}$$

where $\dot{z}$ on the left-hand side is the first order derivative of z with respect to time. The range of the variable is $0 < z < 1$, where the lower limit corresponds to the equilibrium position where $x = 0$, $y = 1/e$, and the upper limit to the amplitude positions given by equation (4), where $y = 1$. At the relativistic energy equal to twice the rest energy, that is $E = 2mc^2$, the energy parameter $e = 2$, and it is opportune to restrict the solution to the energy range given by $1 < e < 2$, that is $0 < e - 1 < 1$, when the square-bracket factors in equation (6) can be expanded in convergent Maclaurin series.

However, expansions in power series are likely to lead to a sufficiently accurate solution only when the parameter $\varepsilon = e - 1$ is much less than unity, which is valid at a sufficiently low energy, and that is assumed in the following.

In the approximation of low energy, the reciprocals of the square-bracket factors in equation (6) can be expanded in power series and multiplied; the ensuing series including the non-expanded factors can be integrated term by term, and the integrals are reducible to elementary functions. The result obtained by retaining terms up to the first order is

$$\sin^{-1}(2z - 1) + (\tfrac{7}{4})\varepsilon(z - z^2)^{1/2} = \pm\, 2q\omega t - \pi/2, \tag{7}$$

where q is a coefficient of order unity and a function of the parameter ε only, and the additional phase term on the right-hand side is due to the inverse sine function at $t = 0$ and $z = 0$; it is assumed that the particle is initially at the equilibrium position with an initial velocity equal to the maximum velocity. Upon taking the sine of both sides of equation (7) and applying the addition formula to the left-hand side, it follows that

$$(2z-1)\cos[(\tfrac{7}{4})\varepsilon(z-z^2)^{1/2}] + [1-(2z-1)^2]^{1/2}\sin[(\tfrac{7}{4})\varepsilon(z-z^2)^{1/2}] = -\cos(2q\omega t), \quad (8)$$

Upon expanding the sine and cosine factors on the left-hand side of equation (8) in power series, the resulting equation to the first order is

$$z + (\tfrac{7}{4})\varepsilon(z-z^2) = \sin^2(q\omega t), \quad (9)$$

which can be solved algebraically as a quadratic equation in the variable z. The solution with minus sign is finite and can be expanded to give

$$z = \sin^2(q\omega t) - (\tfrac{7}{4})\varepsilon \sin^2(q\omega t)\cos^2(q\omega t) + \cdots. \quad (10)$$

The solution for the intermediate variable y follows directly from equation (10); then the reciprocal and the square root expansions give for the original variable

$$x = (2mc^2\varepsilon/k)^{1/2}\{[1-(\tfrac{3}{8})\varepsilon]\sin(q\omega t) + (\tfrac{3}{8})\varepsilon\sin^3(q\omega t)\} + \cdots, \quad (11)$$

where to this order of approximation, $q = 1 - (\frac{3}{8})\varepsilon$. The amplitude factor in equation (11) is the same as the extremum value of x given by equation (4), and the two signs are now incorporated in the sine function. In the limit of very low energy, the second term in the curly bracket expression becomes negligible, the coefficient q approaches unity, the quantity $mc^2\varepsilon$ approaches the classical total energy, and the oscillator becomes simple harmonic.

By use of a basic trigonometric reduction and equation (4), equation (11) can be written in the form

$$x/|x_e| = [1-(\tfrac{3}{32})\varepsilon]\sin(q\omega t) - (\tfrac{3}{32})\varepsilon\sin(3q\omega t) + \cdots, \quad (12)$$

which differs in coefficients from an analogous result obtained by the method of harmonic balance [4].

At an energy when the relativistic effects are significant, the amplitude of oscillations is higher than in the classical case, the parameter q is less than unity and the basic frequency of oscillations is lower than classically, as can be seen from equations (11) and (12), and the basic period of oscillations is greater than $2\pi/\omega$. Since this approximation is valid when the parameter ε is much less than unity, the cases of high energy and the ultrarelativistic limit require further investigation.

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